

On solvable compact Clifford-Klein forms

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Abstract

In the present article we show that there is a large class of homogeneous spaces G/H of reductive type which cannot be a local model for any compact manifold M with solvable fundamental group. Another way of expressing this is: we prove that under certain assumptions, a reductive homogeneous space G/H does not admit a solvable compact Clifford-Klein form. This generalizes the well known non-existence theorem of Benoist for nilpotent Clifford-Klein forms [1]. This generalization works for a particular class of homogeneous spaces determined by a “very regular” embeddings of H into G .

Keywords: *proper actions, homogeneous spaces, Lie groups*

MSC *57S30, 22F30, 22E40*

1 Introduction

Let G be a Lie group and H a closed subgroup of G . In differential geometry, one very often wants to know how strongly a local geometric structure determines global properties of a given manifold. Consider the case of a homogeneous space G/H . In this framework, one can ask a question (see also [10]): *what is a possible fundamental group of a compact manifold M locally modeled on G/H ?* Assume that we are given a homogeneous space G/H of a semisimple non-compact Lie group G and that there exists a discrete subgroup $\Gamma \subset G$ that acts properly and co-compactly on G/H (that is, the double coset $\Gamma \backslash G/H$ is compact). Note that we may assume Γ to be torsion-free. Therefore Γ acts freely on G/H . Thus, $\Gamma \backslash G/H$ has a natural structure of a smooth manifold. Clearly, under some mild assumptions, $\pi_1(\Gamma \backslash G/H) \cong \Gamma$. It follows that our question can be reformulated as follows:

does G/H admit a proper and co-compact action of a discrete $\Gamma \subset G$? If yes, we say that G/H admits a **compact Clifford-Klein form**.

Problems related to Clifford-Klein forms of homogeneous spaces were intensively studied in many research papers. Let us mention those of them [1, 5, 6, 7, 8, 9, 10, 11, 13, 14, 18, 28, 30], which were inspiring for us. The question of the existence of compact Clifford-Klein forms seems to be challenging.

1. The Calabi-Markus phenomenon [11] states that if G is reductive, H is a reductive connected subgroup and $\text{rank}_{\mathbb{R}}(\mathfrak{g}) = \text{rank}_{\mathbb{R}}(\mathfrak{h})$ then only finite groups can act properly discontinuously on G/H . Thus, there does not exist a compact M locally modeled on G/H and any manifold modeled on G/H has a finite fundamental group.
2. Assume that G is reductive and H is reductive and connected. The work [1] yields a condition for G/H to admit non-virtually abelian discrete subgroups (a group is said to be virtually abelian if it has an abelian subgroup of finite index). For instance

$$X_{SL} := SL(2n+1, \mathbb{R})/SL(2n, \mathbb{R}), \quad n > 1,$$

admits a proper action of an infinite discrete subgroup of G , but every such discontinuous subgroup is virtually abelian. Note that G/H admits a compact Clifford-Klein form only if it admits a non-virtually abelian Clifford-Klein forms, by a result of Benoist [1]. Therefore, there are no compact manifolds locally modeled on X_{SL} and every manifold locally modeled on X_{SL} has a virtually abelian fundamental group.

3. On the other hand, if G is semisimple and $K \subset G$ is a maximal compact subgroup then G/K admits a compact Clifford-Klein form (one can take for Γ a co-compact subgroup $\Gamma \in G$). Also, for example, $SO(4, 4n)/SO(3, 4n)$, admits compact Clifford-Klein forms (see Table 3.3 in [7] for a list of all known symmetric homogeneous spaces admitting compact Clifford-Klein forms).
4. Any compact Clifford-Klein form of $GL(n, \mathbb{R}) \ltimes \mathbb{R}^n/GL(n, \mathbb{R})$ is conjectured to be virtually solvable (Auslander's conjecture).
5. A theorem of Benoist [1] shows that *nilpotent groups* cannot yield compact Clifford-Klein forms of semisimple homogeneous spaces. In greater

detail, the following holds: assume that G/H is a non-compact homogeneous space of a semisimple real Lie group. If a nilpotent subgroup $N \subset G$ acts properly on G/H , then $N \backslash G/H$ cannot be compact. Therefore there are no compact manifolds with nilpotent fundamental groups locally modeled on semisimple G/H .

In the present paper we study the following problem: *does there exist a manifold locally modeled on a homogeneous space G/H of reductive type, with a solvable fundamental group?* Note that we understand "locally modeled" as a diffeomorphism $M \cong \Gamma \backslash G/H$, because we are interested in geometric applications (since Γ is a subgroup of G acting on G/H by left translations, invariant geometric structures on G/H descend onto M).

The problem is much more complicated than in the nilpotent case, as there may exist solvable, non-discrete subgroups of G acting properly and co-compactly on G/H . For instance, take $X = SO(4, 4)/SO(3, 4)$. One may prove that

$$SO(4, 4) = SO(3, 4)SO(1, 4)$$

for certain embeddings $SO(3, 4), SO(1, 4) \hookrightarrow SO(4, 4)$ such that $SO(3, 4) \cap SO(1, 4) = SO(3)$ (see Table 2 in [21]). Therefore

$$SO(4, 4)/SO(3, 4) \cong SO(1, 4)/SO(3)$$

and since $SO(3)$ is compact any subgroup of $SO(1, 4)$ acts properly on $SO(1, 4)/SO(3)$ (and thus, on $SO(4, 4)/SO(3, 4)$).

Let $SO(1, 4) = KAN$ be the Iwasawa decomposition of $SO(1, 4)$. Then AN is a solvable subgroup and, since it is closed, it acts properly on X . Moreover AN acts co-compactly on X (this follows from Theorem 2).

In the present article we show that there is a large class of homogeneous spaces G/H of reductive type which cannot be a local model for any compact manifold M .

Definition 1. Let G be a non-compact semisimple real linear Lie group. We will say that a closed subgroup $H \subset G$ is of parabolic type, if H is the semisimple part of the Levi subgroup of some parabolic subgroup $P \subset G$. We will call G/H the homogeneous space of parabolic type.

Example 1. The following pairs (H, G) , $H \subset G$, represent subgroups of parabolic type.

$$(SL(m, \mathbb{R}), SL(n, \mathbb{R})), (Sp(m, \mathbb{R}), Sp(n, \mathbb{R})),$$

$$\begin{aligned}
& (SO^*(2m), SO^*(2n)), (SU(m, m), SU(n, n)); \ 1 < m < n \\
& (SO(k, k), E_6^I), (SL(k+1, \mathbb{R}), E_6^I); \ 1 \leq k \leq 5 \\
& (SU(3, 3), E_6^{II}), (E_6^I, E_7^V), (SO(3, 4), F_4^I), (E_7^V, E_8^{VIII}), (SO(3, 11), E_8^{IX})
\end{aligned}$$

For the notation see Table 9, pages 312-317, [19].

Our main result is the following.

Theorem 1. *Let G/H be a homogeneous space of parabolic type. Assume that a virtually solvable subgroup $\Gamma \subset G$ acts properly on G/H . Then the space $\Gamma \backslash G/H$ is non-compact. Therefore there are no compact manifolds with solvable fundamental groups locally modeled on homogeneous spaces of parabolic type.*

Corollary 1. *The following homogeneous spaces do not admit virtually solvable compact Clifford-Klein forms*

$$SL(n, \mathbb{R})/SL(m, \mathbb{R}), \ E_8^{VIII}/E_7^V, \ n > m > 1.$$

Notice that $SL(n, \mathbb{R})/SL(m, \mathbb{R})$ is an important “test example” in the theory of compact Clifford-Klein forms. Benoist proved in [1] that the homogeneous space $SL(2n+1, \mathbb{R})/SL(2n, \mathbb{R})$, $n > 1$ does not admit compact Clifford-Klein forms at all. The general case $SL(n, \mathbb{R})/SL(m, \mathbb{R})$ is still open and of significant interest (see [1], [13], [14], [16] for partial results).

There is yet another way of looking at the main result of this work. One of the important and challenging problems in the whole area is Kobayashi’s conjecture, which we now describe (see Conjecture 3.3.10 in [7]). Assume that G/H is a homogeneous space of reductive type. We say that G/H admits a standard Clifford-Klein form, if there exists a reductive Lie subgroup $L \subset G$ such that L acts properly on G/H and $L \backslash G/H$ is compact. We will call (G, H, L) a standard triple. The Kobayashi’s conjecture states that for semisimple real Lie groups G and homogeneous spaces G/H of reductive type, the existence of a compact Clifford-Klein form on G/H implies the existence of a standard one. Notice that the Kobayashi’s conjecture indicates that compact Clifford-Klein forms of non-compact homogeneous spaces G/H of reductive type are rare and of a special nature. Our motivation is to eliminate the possibility of obtaining Clifford-Klein forms from double quotients of connected subgroups that are solvable, and, therefore, to obtain further evidence for the conjecture. From this point of view, the theorem of Benoist

[1], which we have already mentioned, shows that *nilpotent groups* cannot yield Clifford-Klein forms. In this paper we show that essentially larger class of solvable groups cannot yield Clifford-Klein forms of homogeneous spaces G/H determined by “regularly embedded” subgroups H .

2 Preliminaries

Throughout this article we use the basics of Lie theory without further explanations closely following [20]. Also we denote the Lie algebras of Lie groups G, H, \dots by the corresponding Gothic letters $\mathfrak{g}, \mathfrak{h}, \dots$ etc. We also use relations between real Lie groups and algebraic groups following [15], [19], [25], [26], [29]. Let G be a real connected linear semisimple Lie group with the Lie algebra \mathfrak{g} and $H \subset G$ a closed connected subgroup with the Lie algebra \mathfrak{h} . The Lie algebra \mathfrak{g} has a Cartan decomposition (and the corresponding Cartan involution)

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

where \mathfrak{k} is a maximal compact Lie subalgebra. Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Note that all such subalgebras are conjugate in \mathfrak{g} . There is a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ (called the split Cartan subalgebra) of the form

$$\mathfrak{t} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}.$$

Then $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}}$, and, therefore, determines the root system Σ of $\mathfrak{g}^{\mathbb{C}}$. The (non-zero) restrictions of $\alpha \in \Sigma$ on \mathfrak{a} form a system of restricted roots Δ (which is an “abstract” root system itself). In this article we will use only restricted roots. Therefore, throughout this paper we will call them “real roots”. Recall that the real root decomposition is given by the formula

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha},$$

where \mathfrak{g}_{α} are the weight subspaces of the adjoint representation $ad \mathfrak{a}$ (\mathfrak{g}_{α} need not to be one-dimensional). Note that we will use the basics of the theory of root systems in the real case.

The Weyl group W of \mathfrak{g} is the finite group of orthogonal transformations of \mathfrak{a} (with respect to the Killing form of \mathfrak{g}) generated by reflections in hy-

perplanes $C_\alpha := \{X \in \mathfrak{a} \mid \alpha(X) = 0\}$ for $\alpha \in \Delta$. The following is well known.

Proposition 1 ([20], Proposition 4.2, Chapter 4). *The group W coincides with the group of transformations induced by automorphisms Ad_k ($k \in N_K(\mathfrak{a})$) and also with the group of transformations induced by automorphisms Ad_g ($g \in N_G(\mathfrak{a})$). Therefore*

$$W \cong N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \cong N_G(\mathfrak{a})/Z_K(\mathfrak{a}).$$

Also, the Weyl chamber \mathfrak{a}^+ is determined by a set of positive roots Δ^+ for Δ .

Definition 2. *Let G be a real semisimple Lie group and H be a closed subgroup. The homogeneous space G/H is of reductive type if there exists a Cartan involution θ of \mathfrak{g} such that $\theta(\mathfrak{h}) = \mathfrak{h}$.*

Notice that if G/H is of reductive type then H is a reductive Lie group. Also, we use a relation between Lie groups and linear algebraic \mathbb{R} -groups (see [15]). If $\mathbf{G} \subset GL(n, \mathbb{C})$ is an algebraic \mathbb{R} -group, then $G = \mathbf{G}_{\mathbb{R}} = \mathbf{G} \cap GL(n, \mathbb{R})$ is a Lie group with a finite number of connected components.

Let X be a Hausdorff topological space and Γ a topological group acting on X . We say that an action Γ on X is proper, if for any compact subset $S \subset X$ the set

$$\{\gamma \in \Gamma \mid \gamma(S) \cap S \neq \emptyset\}$$

is compact. In particular, in this article we consider the proper actions of $\Gamma \subset G$ on $X = G/H$ by left translations. It easily follows from the definition that if a closed connected Lie subgroup $L \subset G$ acts properly on G/H then any orbit of L is closed.

We will also need the following results on proper actions. For a connected Lie group J define $d(J) := \dim J - \dim K_J$, where K_J is a maximal compact subgroup of J .

Theorem 2 ([11], Theorem 4.7 and [18], Theorem 3.4). *Let $A \subset G$ and $B \subset G$ be closed connected subgroups of G . If A acts properly on G/B then*

$$A \backslash G/B \text{ is compact iff } d(G) = d(A) + d(B).$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and let Δ^+ be a subset of positive roots of the real root system Δ with respect to a fixed $\mathfrak{a} \subset \mathfrak{p}$. Set

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

One can easily see that \mathfrak{n} is a real nilpotent subalgebra of \mathfrak{g} . Also, $\mathfrak{a} + \mathfrak{n}$ is a solvable subalgebra of \mathfrak{g} . One obtains the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n},$$

where \mathfrak{k} is a maximal compact subalgebra of \mathfrak{g} . It follows from Definition 2, that if G/H is of reductive type, then \mathfrak{h} admits a Cartan decomposition compatible with that of \mathfrak{g} :

$$\mathfrak{h} = \mathfrak{k}_h + \mathfrak{p}_h, \quad \mathfrak{k}_h = \mathfrak{k} \cap \mathfrak{h}, \quad \mathfrak{p}_h = \mathfrak{p} \cap \mathfrak{h}.$$

In the same way, \mathfrak{h} admits a *compatible Iwasawa decomposition*

$$\mathfrak{h} = \mathfrak{k}_h + \mathfrak{a}_h + \mathfrak{n}_h, \quad \mathfrak{n}_h = \mathfrak{n} \cap \mathfrak{h}, \quad \mathfrak{a}_h = \mathfrak{a} \cap \mathfrak{h}.$$

Now let G be a connected semisimple Lie group whose Lie algebra is \mathfrak{g} . There exists a connected compact Lie subgroup $K \subset G$ and simply connected Lie subgroups A and N whose Lie algebras are \mathfrak{a} and \mathfrak{n} such that

$$G = K \cdot A \cdot N$$

is a topological decomposition into a direct product of subgroups. This decomposition is the (global) Iwasawa decomposition of G . In the same manner, we obtain the *compatible* global Iwasawa decomposition of H : $H = K_h \cdot A_h \cdot N_h$ (the meaning of the symbols is clear). We need also one more decomposition, the Cartan decomposition, $G = K \cdot A \cdot K$. Consider G as a semisimple group of \mathbb{R} -points of an algebraic \mathbb{R} -group \mathbf{G} , and A as a subgroup of \mathbb{R} -points of a maximal algebraic torus \mathbf{A} . Then one can use the root system $\Delta = \Delta(\mathbf{A}, \mathbf{G})$ of \mathbf{G} with respect to \mathbf{A} and define the global Weyl chamber

$$A^+ = \{a \in A \mid \chi(a) > 0, \text{ for any } \chi \in \Delta^+\}.$$

For more details we refer to [1]. Note that we use the same symbol to denote the root systems for \mathbf{G} and \mathfrak{g} . This easily yields the decomposition

$$G = K \cdot A^+ \cdot K$$

(which is also called the Cartan decomposition). Thus, for any element $g \in G$ there is an element $a_g \in A^+$ such that $g \in K \cdot a_g \cdot K$. This element is unique, hence there is a well defined map $\mu : G \rightarrow A^+$ given by the formula

$$\mu(g) = a_g,$$

called the Cartan projection. The function μ is continuous and proper (that is, the preimage of a compact set is compact). Also, using the diffeomorphism $\log : A^+ \rightarrow \mathfrak{a}^+$ one obtains a proper map $\mu : G \rightarrow \mathfrak{a}^+$, where $\mathfrak{a}^+ \subset \mathfrak{a}$ is a closed Weyl chamber in \mathfrak{a} . Note that we will denote both maps by the same letter, because we will use it only as a map with \mathfrak{a}^+ as a target. We will need the following characterization of a properness of an action of a subgroup on a homogeneous space.

Theorem 3 ([1], Corollary 5.2 and [8], Theorem 1.1). *Let $A, B \subset G$ be closed connected subgroups of G and μ the Cartan projection in G . The subgroup A acts properly on G/B if and only if*

$$\mu(A) \cap (\mu(B) + C)$$

is bounded for every compact subset $C \subset \mathfrak{a}$.

In this paper we are following an approach of Toshiyuki Kobayashi [11]. The latter is based on the following observation.

Proposition 2 ([11], Lemma 2.3). *Let a real Lie group G act on a locally compact Hausdorff space X and Γ be a uniform lattice in G . Then*

1. *The G -action on X is proper iff the Γ -action on X is proper.*
2. *$G \setminus X$ is compact iff $\Gamma \setminus X$ is proper.*

Also, in the proof of the main result we will use the fact that homogeneous spaces G/H of reductive type and of maximal real rank cannot admit proper actions of non virtually abelian discrete subgroups, and, hence, cannot admit compact Clifford-Klein forms. This fact is called the Calabi-Markus phenomenon.

Theorem 4 ([11], Corollary 4.4). *Let G/H be a homogeneous space of reductive type. If $\text{rank}_{\mathbb{R}} G = \text{rank}_{\mathbb{R}} H$, then only finite groups can act properly on G/H . In particular, such G/H cannot have compact Clifford-Klein forms.*

In the proof of Theorem 1 we will need the Jacobson-Morozov theorem (see [3], Theorem 9.2.1). We say that a triple (H, X, Y) of vectors in \mathfrak{g} is an \mathfrak{sl}_2 -triple, if

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H.$$

Theorem 5 (Jacobson-Morozov). *Let \mathfrak{g} be a real semisimple Lie algebra and X be a non-zero nilpotent element. Then there exists an \mathfrak{sl}_2 -triple (H, X, Y) in \mathfrak{g} .*

Since we consider real Lie algebras, we use the following definition of a parabolic subgroup: it is parabolic, if its Lie subalgebra \mathfrak{q} is parabolic in the following sense: the complexification \mathfrak{q}^c is a parabolic Lie subalgebra in the complexification \mathfrak{g}^c of \mathfrak{g} . Note that although we define \mathfrak{q} in terms of Lie algebras over \mathbb{C} , it admits a complete description in terms of the real root system of \mathfrak{g} . We will present the details in Section 3. Also, we refer to [20] (Chapter 6, Section 1.5) and [22].

3 Proof of Theorem 1

3.1 Zariski closures and syndetic hulls

One of the important tools in the proof of Theorem 1 is the following result from [4].

Theorem 6 ([4], Section 1.6). *Let V be a finite-dimensional real vector space and G a virtually solvable subgroup of $GL(V)$. Then there exists at least one closed virtually solvable subgroup $H \subset GL(V)$ containing G such that:*

1. *H has finitely many components and each component meets G ;*
2. *(syndeticity) there exists a compact set $K \subset H$ such that $H = K \cdot G$;*
3. *H and G have the same Zariski closure in $GL(V)$.*

Assume now that we are given a homogeneous space of a reductive type G/H , and that G is connected and linear, thus, $G \subset GL(V)$. Assume that Γ is a solvable discrete subgroup of G acting properly and co-compactly on G/H . Consider the Zariski closure $L = \bar{\Gamma}$. Apply Theorem 6 to Γ (instead of G). We obtain that there exists a subgroup $B \subset GL(V)$ such that $B \supset \Gamma$ and $\bar{B} = \bar{B} = L$ (that is, we have B instead of H in Theorem 6). Since L is the

Zariski closure of Γ , it is also solvable. Therefore, since $\bar{B} = L$, we obtain a (virtually) solvable subgroup B such that $\Gamma \setminus B$ is compact. Consider the connected component B_0 . Note that B has only finite number of connected components, hence B_0 is a connected Lie subgroup in G . Clearly, B_0 must be solvable. Since the Lie subgroup B contains a uniform lattice $B \subset G$, so does B_0 . This is a consequence (for example) of the following facts (see [29], Chapter 1, Sections 4.1 and 4.2):

- For a closed subgroup S in a Lie group R a discrete subgroup $\Gamma \subset R$ has the property $\Gamma \cap S$ is a uniform lattice in S if and only if $S \cdot \Gamma$ is closed in R ,
- the following two conditions are equivalent: a) S is a closed Lie subgroup such that $S \cdot \Gamma$ is closed, b) the map $\varphi : \Gamma/S \cap \Gamma \rightarrow R/S$ which is a restriction of the projection $R \rightarrow R/S$ has a discrete image.

Applying the latter to $R = B$ and $S = B_0$, one obtains that $B_0 \cap \Gamma$ is a uniform lattice in B_0 .

Definition 3. The subgroup $B \subset G$ constructed in Theorem 6 will be called the *syndetic hull* of Γ .

We summarize our consideration as follows.

Lemma 1. *Assume that G/H is a homogeneous space of reductive type and that G is a connected real linear Lie group. If a discrete solvable subgroup $\Gamma \subset G$ acts properly and co-compactly on G/H , then it admits a syndetic hull B , which is a connected solvable Lie subgroup admitting a uniform lattice Γ . The syndetic hull B acts properly and co-compactly on G/H .*

Note that the last claim of the lemma follows from Proposition 2.

3.2 Preliminaries needed in the proof of Theorem 1

We will use the following Lemma and Theorem in the proof of the main result.

Lemma 2 ([10], Lemma 1.3). *Let G_1, G_2 be locally compact groups and $L_1, H_1 \subset G_1$, $L_2, H_2 \subset G_2$ be closed subgroups. Assume that $f : G_1 \rightarrow G_2$ is a continuous homomorphism such that $f(L_1) \subset L_2$, $f(H_1) \subset H_2$, $f(L_1)$ is closed in G_2 and $L_1 \cap \text{Ker } f$ is compact. Then if the action of L_2 on G_2/H_2 is proper then the action of L_1 on G_1/H_1 is proper.*

Theorem 7 ([28], Theorem 6.2). *Let M and N be connected subgroups of a connected, simply connected, solvable Lie group S . If $M \backslash S/N$ is compact, and every orbit of M is closed, then $S = MN$.*

Following Onishchik [24] we introduce the notion of the *factorizations of Lie groups and Lie algebras*.

Definition 4. We say that a triple (G, H, L) of Lie groups is a factorization, if H and L are Lie subgroups of G and $G = H \cdot L$. In the same way, a triple of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ is called a factorization, if \mathfrak{h} and \mathfrak{l} are Lie subalgebras of \mathfrak{g} , and $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$.

We will need the following straightforward result.

Proposition 3 ([24], Corollary on p. 88). *If a triple of Lie groups is a factorization, so is the triple of their Lie algebras.*

3.3 Inclusion $B \subset TUC_N(T)$

Use the notation from the previous subsection. In this subsection we don't use the fact that B is a syndetic hull of Γ . Instead of that, we consider the following. Assume that we are given a reductive homogeneous space G/H and a closed connected solvable subgroup $B \subset AN$ acting properly and co-compactly on G/H . Let $T = A \cap (BN)$, $U = B \cap N$ and denote by $C_N(T)$ the centralizer of T in N . Consider the Zariski closure $\bar{\Gamma} = \bar{B} = L$ of Γ . Denote the Lie algebras of $B, \bar{B} = L, \bar{U}, U, \bar{T}, T, C_N(\bar{T}), C_N(T)$ by

$$\mathfrak{b}, \mathfrak{l}, \bar{\mathfrak{u}}, \mathfrak{u}, \bar{\mathfrak{t}}, \mathfrak{t}, \mathfrak{c}, \mathfrak{c}_t,$$

respectively. Following [17], we say that B is *compatible with A* , if $B \subset TUC_N(T)$.

Lemma 3 ([17], Lemma 2.3). *If B is a closed connected subgroup of AN , then it is conjugate, via an element of N , to a subgroup which is compatible with A .*

Thus, in our considerations we will always assume that

$$B \subset TUC_N(T).$$

Lemma 3 implies also the following.

Lemma 4. *The following holds: $\mathfrak{u} + \mathfrak{c} = \bar{\mathfrak{u}}$, $\mathfrak{t} \subset \bar{\mathfrak{t}}$, $[\mathfrak{t}, \mathfrak{u}] \subset \mathfrak{u}$.*

Proof. The proof of this lemma is contained in the proof of Lemma 3 ([17], Lemma 2.3). Therefore, we reproduce it for the convenience of the reader. Let \bar{B} be the identity component of the Zariski closure of B , and write $\bar{B} = \bar{T} \rtimes \bar{U}$, where \bar{U} is a subgroup of N and \bar{T} is conjugate, via element of N , to a subgroup of A . Here we use the well-known fact ([2], Theorem 10.6 (4)) that a real connected solvable algebraic group L is a semidirect product

$$L = \bar{T} \rtimes \bar{U}$$

of a torus, and a unipotent subgroup \bar{U} . We may assume that $\bar{T} \subset A$ (taking a conjugate, if necessary). Let $U = B \cap N$. It is proved in [27] that $[\bar{B}, \bar{B}] \subset B \cap N$, which implies $Ad_G \bar{T}(\bar{\mathfrak{u}}) \subset \mathfrak{u}$. Also, $\bar{T} \subset A$. Clearly, the subalgebra $\bar{\mathfrak{u}}$ is $Ad_G(\bar{T})$ -invariant, and the adjoint action of \bar{T} on $\bar{\mathfrak{u}}$ is completely reducible. It follows that there is a subspace $\mathfrak{c} \subset \bar{\mathfrak{u}}$ such that

$$Ad_G(\bar{T})(\mathfrak{c}) = 0, \text{ and } \mathfrak{u} + \mathfrak{c} = \bar{\mathfrak{u}}.$$

Therefore, $UC_N(\bar{T}) = \bar{U}$, so $\bar{B} = \bar{T}UC_N(\bar{T})$. Let $\pi : AN \rightarrow A$ be the projection with the kernel N , and let $T = \pi(B)$. We get

$$T = \pi(B) \subset \pi(\bar{B}) = \bar{T} \Rightarrow C_N(T) \supset C_N(\bar{T}).$$

For any $b \in B$, there exist $t \in \bar{T}, u \in U$ and $c \in C_N(\bar{T})$ such that $b = tuc$. But $uc \in N$, hence $t = \pi(b) \in T$, and, because $C_N(T) \supset C_N(\bar{T})$, we obtain $c \in C_N(T)$. Therefore, $b \in TUC_N(T)$. Finally, $[\bar{\mathfrak{t}}, \bar{\mathfrak{u}}] \subset \bar{\mathfrak{u}}$. Now, it follows from Lemma 4 that $[\mathfrak{t}, \mathfrak{u}] \subset \mathfrak{u}$. The proof of both Lemma 3 and Lemma 4 is complete. \square

3.4 Non-unimodularity of B

Theorem 8. *Let $B \subset AN$ be a compatible subgroup acting properly and co-compactly on a homogeneous space G/H of a parabolic type. Assume that $\text{rank}_{\mathbb{R}} G > \text{rank}_{\mathbb{R}} H$. Then B cannot be unimodular.*

Proof. Consider the inclusion $f : AN \hookrightarrow G$ and put $L_1 = A_h N_h$, $L_2 = AN$, $H_1 = A_h N_h$, $H_2 = H$. Apply Lemma 2. This shows that B acts properly on $AN/A_h N_h$. Also, B acts co-compactly on $AN/A_h N_h$. The latter easily

follows from Theorem 2. Indeed, $d(B) + d(H) = d(G)$, since B acts co-compactly on G/H . But $d(H) = d(A_h N_h)$, and $d(AN) = d(G)$. Hence, $d(B) + d(A_h N_h) = d(AN)$, and Theorem 2 applies. Therefore, every orbit of this action is closed. Applying Theorem 7 we obtain a decomposition

$$AN = B(A_h N_h) = LA_h N_h. \quad (1)$$

Applying Proposition 3 and the equality $\mathfrak{l} = \bar{\mathfrak{t}} + \bar{\mathfrak{u}}$ we get

$$\mathfrak{a} + \mathfrak{n} = (\bar{\mathfrak{t}} + \bar{\mathfrak{u}}) + (\mathfrak{a}_h + \mathfrak{n}_h).$$

Since $\bar{\mathfrak{t}} \subset \mathfrak{a}$ and $\bar{\mathfrak{u}} \subset \mathfrak{n}$ (see the proof of Lemma 4), one obtains

$$\mathfrak{n} = \mathfrak{n}_h + \bar{\mathfrak{u}}. \quad (2)$$

By definition, $U \subset B$. Therefore, it is easy to see that one can write

$$B \subset U[B \cap TC_N(T)].$$

The latter equality implies

$$\mathfrak{b} \subset \mathfrak{u} + \mathfrak{t} + \mathfrak{c}_t. \quad (3)$$

Therefore, (1) and (3) imply

$$(\mathfrak{a} + \mathfrak{n}) = \mathfrak{b} + (\mathfrak{a}_h + \mathfrak{n}_h) = (\mathfrak{t} + \mathfrak{u} + \mathfrak{c}_t) + (\mathfrak{a}_h + \mathfrak{n}_h)$$

and since $\mathfrak{c}_t \subset \mathfrak{n}$ we have

$$\mathfrak{t} + \mathfrak{a}_h = \mathfrak{a}. \quad (4)$$

Consider the decomposition

$$(\mathfrak{a} + \mathfrak{n}) = \mathfrak{a} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Since \mathfrak{h} is a semisimple part in some parabolic subalgebra of \mathfrak{g} it follows that \mathfrak{h} admits an Iwasawa decomposition (see [20], Section 1.5, Chapter 6) $\mathfrak{h} = \mathfrak{k}_h + \mathfrak{a}_h + \mathfrak{n}_h$ such that $\mathfrak{a}_h \subset \mathfrak{a}$ and

$$(\mathfrak{a}_h + \mathfrak{n}_h) = \mathfrak{a}_h + \sum_{\alpha \in \Delta_h^+} \mathfrak{g}_\alpha.$$

for a subset of positive roots Δ_h^+ of $\Delta_h \subset \Delta$, where Δ_h is the root system of \mathfrak{h} and Δ is the root system of \mathfrak{g} . Moreover, there exists $x_h \in \mathfrak{a}$ so that

$$\Delta_h = \{\alpha \in \Delta \mid \alpha(x_h) = 0\}.$$

Define $\Delta_x^+ \subset \Delta$ by $\Delta_x^+ := \{\alpha \in \Delta \mid \alpha(x_h) > 0\}$ and put

$$\mathfrak{n}_x := \sum_{\alpha \in \Delta_x^+} \mathfrak{g}_\alpha.$$

Therefore (since $\Delta_x^+ \cup \Delta_h^+ = \Delta^+$) we have a decomposition

$$\mathfrak{n} = \mathfrak{n}_h + \mathfrak{n}_x. \quad (5)$$

Since \mathfrak{n}_h is given by some subset of root spaces of \mathfrak{g} , it follows that

$$[\mathfrak{t}, \mathfrak{n}_h] \subset \mathfrak{n}_h.$$

Thus for any $Y \in \mathfrak{t}$ the decomposition (5) is ad_Y -invariant. Now we will get one more ad_Y -invariant decomposition. By (2), and the decomposition $\bar{\mathfrak{u}} = \mathfrak{u} + \mathfrak{c}$ combined with the obvious inclusion $\mathfrak{c} \subset \mathfrak{c}_t$ we obtain

$$\mathfrak{n} = \mathfrak{n}_h + \mathfrak{u} + \mathfrak{c}_t. \quad (6)$$

The decomposition (6) is ad_Y -invariant, although not direct. We understand ad_Y -invariance in the sense that subspaces \mathfrak{n}_h and $\mathfrak{u} + \mathfrak{c}_t$ are ad_Y -invariant, which follows from $[\mathfrak{t}, \mathfrak{u}] \subset \mathfrak{u}$ (Lemma 4), the assumptions on \mathfrak{h} and $[\mathfrak{t}, \mathfrak{c}_t] = 0$. Note that $\mathfrak{u} \cap \mathfrak{n}_h = \{0\}$, since $U \subset B$ and B acts properly on G/H . In greater detail, we argue as follows. If $\mathfrak{n}_h \cap \mathfrak{u}$ is not trivial, it contains $\mathbb{R}X$ for some nonzero $X \in \mathfrak{n}$. It follows from the Jacobson-Morozov Theorem (Theorem 5) that there exists an \mathfrak{sl}_2 -triple that contains X . Since the Cartan projection of \tilde{N} equals the Cartan projection of \tilde{A} for any semisimple, connected Lie group S with the Iwasawa decomposition $S = \tilde{K}\tilde{A}\tilde{N}$ (see Theorem 5.1 in [12]), we see that $\mu(H) \cap \mu(B)$ is not bounded, a contradiction.

Let $\mathfrak{m} = (\mathfrak{n}_h + \mathfrak{u}) \cap \mathfrak{c}_t$. It is straightforward to see that this subspace is ad_Y -invariant. Writing down an ad_Y -invariant decomposition $\mathfrak{c}_t = \mathfrak{m} \oplus \mathfrak{m}'$ one obtains one more ad_Y -invariant (direct) decomposition

$$\mathfrak{n} = \mathfrak{n}_h \oplus \mathfrak{u} \oplus \mathfrak{m}'. \quad (7)$$

Finally, (5) and (7) together yield, for any $Y \in \mathfrak{t}$, two decompositions of \mathfrak{n} into invariant subspaces of the endomorphism $ad_Y : \mathfrak{n} \rightarrow \mathfrak{n}$, namely:

$$\mathfrak{n} = \mathfrak{n}_h + \mathfrak{n}_x \text{ and } \mathfrak{n} = \mathfrak{n}_h + \mathfrak{u} + \mathfrak{m}'. \quad (8)$$

We will use (8) as follows. Notice that for any $Y \in \mathfrak{t}$ one has $ad_Y(\mathfrak{b}) \subset \mathfrak{b}$. Indeed,

$$ad_Y(\mathfrak{t} + \mathfrak{u} + \mathfrak{c}_t) \subset \mathfrak{u} \subset \mathfrak{b}.$$

From this and (8), as well as the fact that the trace does not depend on the basis we obtain

$$Tr((ad_Y)|_{\mathfrak{b}}) = Tr((ad_Y)|_{\mathfrak{u}}) = Tr((ad_Y)|_{\mathfrak{n}_x}). \quad (9)$$

Recall that we assume that \mathfrak{b} must be unimodular. We will show that there exists $Z = Y + X \in \mathfrak{b}$ such that:

- $Y \in \mathfrak{t}$, $X \in \mathfrak{c}_t \subset \mathfrak{n}$,
- $Tr((ad_Y)|_{\mathfrak{n}_x}) \neq 0$.

Since ad_X is nilpotent, the latter yields

$$Tr((ad_Z)|_{\mathfrak{b}}) = Tr((ad_Y)|_{\mathfrak{b}}) = Tr((ad_Y)|_{\mathfrak{n}_x}) \neq 0.$$

Therefore, for such $Z \in \mathfrak{b}$, $tr(ad_Z|_{\mathfrak{b}}) \neq 0$. This shows that \mathfrak{b} cannot be unimodular, and we arrive at a contradiction. It remains to show the existence of Z . We complete the argument in the four steps below.

Step 1. Let W_g be the Weyl group of \mathfrak{g} . For $Y \in \mathfrak{a}$, $w \in W_g$ and $\alpha \in \Delta$ we have $\alpha(Y) = (w\alpha)(wY)$. Indeed, since W_g acts on \mathfrak{a} by orthogonal transformations (with respect to the Killing form \mathcal{K} of \mathfrak{g}) we obtain

$$\alpha(Y) = \mathcal{K}(H_\alpha, Y) = \mathcal{K}(wH_\alpha, wY) = \mathcal{K}(H_{w\alpha}, wY) = w\alpha(wY),$$

where $H_\alpha \in \mathfrak{a}$ denotes the root vector of α .

Step 2. Let W_h be the Weyl group of \mathfrak{h} . Notice that W_h is a subgroup of the Weyl group W_g of \mathfrak{g} (since $\Delta_h \subset \Delta$) and for any $w \in W_h$, we have

$$w(\Delta_x^+) = \Delta_x^+.$$

To see that, notice first, that for any $\alpha \in \Delta_h$

$$\alpha(x_h) = 0 \Leftrightarrow s_\alpha(x_h) = x_h,$$

where $s_\alpha \in W_h$ denotes the reflection induced by α . Now it follows from Step 1 that

$$0 < \alpha(x_h) = w\alpha(wx_h) = w\alpha(x_h).$$

Therefore $\alpha \in \Delta_x^+$ iff $w\alpha \in \Delta_x^+$.

Step 3. Take $\xi := \sum_{\alpha \in \Delta_x^+} a_\alpha \alpha$, where $a_\alpha = \dim(\mathfrak{g}_\alpha)$. Take $w \in W_g$. By Proposition 1, W is isomorphic to $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ and thus there exists $k \in K$ such that $w = \text{Ad}k|_{\mathfrak{a}}$. Since

$$\begin{aligned} X \in \mathfrak{g}_\alpha &\Leftrightarrow \forall_{H \in \mathfrak{a}} [H, X] = \alpha(H)X \Leftrightarrow \forall_{H \in \mathfrak{a}} \text{Ad}k([H, X]) = \text{Ad}k(\alpha(H)X) \\ &\Leftrightarrow \forall_{H \in \mathfrak{a}} [wH, \text{Ad}k(X)] = \alpha(H)\text{Ad}k(X) \Leftrightarrow (**) \end{aligned}$$

It follows from Step 1 that

$$\begin{aligned} (**) &\Leftrightarrow \forall_{H \in \mathfrak{a}} [wH, \text{Ad}k(X)] = w\alpha(wH)\text{Ad}k(X) \\ &\Leftrightarrow \forall_{H \in \mathfrak{a}} [H, \text{Ad}k(X)] = w\alpha(H)\text{Ad}k(X) \Leftrightarrow \text{Ad}k(X) \in \mathfrak{g}_{w\alpha}. \end{aligned}$$

Therefore

$$\dim(\mathfrak{g}_\alpha) = \dim(\text{Ad}k(\mathfrak{g}_\alpha)) = \dim(\mathfrak{g}_{w\alpha}).$$

This implies that for $w \in W_h$ we have $w\xi = \xi$, since $w(\Delta_x^+) = \Delta_x^+$. Therefore ξ^* (that is, the vector dual to ξ with respect to the Killing form of \mathfrak{g}) is perpendicular to \mathfrak{a}_h , because \mathfrak{a}_h is spanned by $\{\alpha^* \mid \alpha \in \Delta_h\}$.

Step 4. Note that $\mathfrak{a} = \mathfrak{t} + \mathfrak{a}_h$ and $\mathfrak{a}_h \neq \mathfrak{a}$ (this is the assumption $\text{rank}_{\mathbb{R}} G > \text{rank}_{\mathbb{R}} H$, compare the Calabi-Markus phenomenon, that is, Theorem 4)). It follows from Step 3 that there exists $Y \in \mathfrak{t}$ that is not perpendicular to ξ^* . Moreover

$$\text{Tr}(\text{ad}Y|_{\mathfrak{n}_x}) = \xi(Y)$$

is obviously nonzero.

Finally, it remains to prove that having $Y \in \mathfrak{t}$ with the property $\text{Tr}(\text{ad}Y|_{\mathfrak{n}_x}) \neq 0$ there exists $Z = Y + X \in \mathfrak{b}$ with nilpotent X . Note that AN is a semidirect product of A and N , therefore, the projection $\pi : AN \rightarrow A$ onto the first factor, is a homomorphism. Note that $\pi(B) = T$ (see the proof of Lemma 4). It follows that $d\pi : \mathfrak{a} + \mathfrak{n} \rightarrow \mathfrak{a}$ is a projection as well. Hence, $Y = d\pi(Y + X)$, where $X \in \mathfrak{n}$, as required. The proof is complete. \square

3.5 The property $B \subset AN$ and completion of proof

In general, for the syndetic hull, the inclusion $B \subset AN$ does not hold. However, we will show that we may assume this in our context. To do this, we need some preparations. Let G be a real semisimple and connected Lie group with an Iwasawa decomposition $G = KAN$. Recall that an element $g \in G$ is called

- hyperbolic, if g is conjugate to an element in A ,
- unipotent, if g is conjugate to an element in N ,
- elliptic, if g is conjugate to an element in K .

In what follows we will use the following facts from [5] (see Subsections 10.2-10.9 in this paper).

Lemma 5 ([5]). *Each $g \in G$ has a unique decomposition*

$$g = auc$$

where a is hyperbolic, u is unipotent and c is elliptic. Moreover:

- a, u and c commute,
- $a, u, c \in \overline{\langle g \rangle}$, where $\overline{\langle g \rangle}$ denotes the Zariski closure of $\langle g \rangle$.

The decomposition $g = auc$ is called *the real Jordan decomposition*. In our context, a real algebraic group T is a torus, if T is abelian and Zariski connected, and every element of T is semisimple. A torus T is \mathbb{R} -split, if every element of T is diagonalizable.

Lemma 6 ([28]). *Let T be a torus. If T_{split} is the maximal \mathbb{R} -split subtorus of T , and T_{cpt} is the maximal compact subtorus of T , then*

$$T = T_{split} \cdot T_{cpt}$$

and $T_{split} \cap T_{cpt}$ is finite.

We will also need the following well known fact (see [29]).

Proposition 4. *Let Γ be a (co-compact) lattice in a locally compact topological group L , and L_1 be a normal subgroup. Let $\pi : L \rightarrow L/L_1$ be the natural projection onto the quotient group. Then $\Gamma \cap L_1$ is a (co-compact) lattice in L_1 if and only if $\pi(\Gamma) \subset L/L_1$ is a (co-compact lattice) in L/L_1 .*

Again, $L = \bar{\Gamma} = \bar{B}$ is real algebraic, and, hence it is a semidirect product

$$L = \bar{T} \ltimes \bar{U}$$

of a torus, and a unipotent subgroup \bar{U} . By Lemma 6, $\bar{T} = T_{split} \cdot T_{cpt}$, and the latter subtori have finite intersection. Therefore, any $l \in L$ has the unique real Jordan decomposition

$$l = t_a t_c u, \quad t_a \in T_{split}, t_c \in T_{cpt}, \quad u \in U.$$

Clearly, $L_1 = T_{split} \ltimes U$ is normal in L , and L/L_1 is compact. Also, by Theorem 6 B is closed. Consider $B \cap L_1$. It follows that $B \cap L_1$ is normal in B and $B/B \cap L_1$ is a closed subgroup in the (Lie) group L/L_1 . Therefore $B/B \cap L_1$ is compact. Referring to Proposition 4 we conclude that $\Gamma \cap (B \cap L_1)$ is a lattice. Also, since $B \cap L_1 = B'$ is co-compact in B , we may assume that B' acts properly and co-compactly on G/H , and, therefore, so does $\Gamma' = B' \cap \Gamma$. It follows that without loss of generality we may assume the following:

$$L = T_{split} \ltimes \bar{U}.$$

In the latter case we have $l = ua$ for each element $l \in L$, and $u, a \in L$ (by Lemma 5, since L is Zariski closed). Then, T_{split} is contained in some maximal split torus \hat{T} of G , that is, in some subgroup conjugate to A . Replacing L by a conjugate, we may assume that $T_{split} \subset A$. In other words we know that $L \cap A$ is a maximal split torus in L . Using this we can prove that

$$L = T_{split}(L \cap N) \tag{*}$$

(that is, $\bar{U} = L \cap N$) simply repeating the proof of Lemma 10.4 by Iozzi and Witte Morris in [5]. For the convenience of the reader we repeat their argument. Given $l \in L$ we have $l = au$, and a belongs to a split torus. It is known (from the general theory of solvable algebraic groups) that all maximal split tori of L are conjugate via an element of $L \cap N$, so there is some $x \in L \cap N$ such that $x^{-1}ax \in A$. Then $\langle T_{split}, x^{-1}ax \rangle$ being a subgroup of A , is a split torus. Thus, the maximality of T_{split} implies that $x^{-1}ax \in T_{split}$. Then, for $t = x^{-1}ax$ one obtains

$$l = au = xtx^{-1}u = t(t^{-1}xt)x^{-1}u \in T_{split}(L \cap N).$$

We conclude that $L = T_{split}(L \cap N)$, as required. It follows that $B \subset L \subset AN$. Now we complete the proof applying Theorem 8.

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